# Sec 1.6 Substitution methods

Goal: Change new types of ODEs, into separable or a $1^{st}$ order linear eq.

Ex

Solve ODE $\left\{ \frac{dy}{dx} = (4x+y)^2 \right\}$ $\leftarrow \cancel{16}x^2 + 8xy + y^2$

Set $v = 4x+y$

Isolate $\boxed{y = v - 4x}$ $\longrightarrow \left\{ \frac{dy}{dx} = v^2 \right\}$

Impl. Diff: $\boxed{\frac{dy}{dx} = \frac{dv}{dx} - 4}$

Sub: $\frac{dy}{dx} = (4x+y)^2 \longrightarrow \frac{dv}{dx} - 4 = v^2$

$$\longrightarrow \boxed{\frac{dv}{dx} = v^2 + 4} \quad \text{(separable)}$$

Solve for v: $\int \frac{dv}{v^2+4} = \int dx \longrightarrow \frac{1}{2}\int \frac{2\,dv}{v^2+2^2} = x + C$

$$\longrightarrow \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + C$$

$$\tan^{-1}\left(\frac{v}{2}\right) = 2x + C$$

$$v = 2\tan(2x + C)$$

"unsub?" $\boxed{y = v - 4x = 2\tan(2x + C) - 4x}$

---

★ For eqs of the form $y' = f(ax+by+c)$, use substitution $v = ax+by+c$.

This makes ODE into a separable eq

**Definition** An equation is "**homogeneous**" if:
Changing $\begin{matrix} x \to cx \\ y \to cy \end{matrix}$ and then simplifying yields the original equation.

**Ex**: Find an implicit solution for $\dfrac{dy}{dx} = \dfrac{2x-y}{x+7y}$.

★ For homogeneous eqs, use substitution ~~#~~ $v = y/x$ (or $y = vx$), to change eq into a separable eq

$$\frac{dy}{dx} = \frac{2x-y}{x+7y} \longrightarrow \frac{d(cy)}{d(cx)} = \frac{2(cx)-cy}{cx+7cy}$$

$$\longrightarrow \frac{\cancel{c}\cdot dy}{\cancel{c}\cdot dx} = \frac{\cancel{c}(2x-y)}{\cancel{c}(x+7y)}$$ ✓ homogeneous.

**set** $v = y/x \longrightarrow \boxed{y = vx}$

**Impl. diff** $\Longrightarrow \boxed{\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}}$

**sub**: $\dfrac{dy}{dx} = \dfrac{2x-y}{x+7y} \longrightarrow v + x\dfrac{dv}{dx} = \dfrac{2\cancel{x} - v\cancel{x}}{\cancel{x} + 7v\cancel{x}}$

$$\longrightarrow x\frac{dv}{dx} = \frac{2-v}{1+7v} - v$$

$$\left\{ x\frac{dv}{dx} = \frac{2-2v-7v^2}{1+7v} \right\} \text{ (separable)}$$

Read rest of example in notes (Next page)

Continuing with $\left\{ x\frac{dv}{dx} = \frac{2-2v-7v^2}{1+7v} \right\}$:

Separable $\longrightarrow$ $\displaystyle\int \frac{(1+7v)\,dv}{2-2v-7v^2} = \int \frac{dx}{x}$

$u = 2-2v-7v^2$

$du = (-2-14v)\,dv$

$du = -2(1+7v)\,dv$

$$\downarrow \qquad\qquad \downarrow$$

$$-\frac{1}{2}\int \frac{du}{u}$$

$$-\frac{1}{2}\ln|u| + C$$

$$\Rightarrow -\frac{1}{2}\ln|2-2v-7v^2| = \ln|x| + C$$

$$\ln|2-2v-7v^2| = -2\ln|x| + C$$

$$\Longrightarrow |2-2v-7v^2| = |x|^{-2}\cdot e^C$$

$$2-2v-7v^2 = \underbrace{\boxed{\pm e^C}}_{\text{“}C\text{”}}\cdot\frac{1}{x^2} \qquad \left(\frac{1}{|x|^2} = \frac{1}{x^2} \text{ b/c } x^2 \geq 0 \text{ always}\right)$$

so $\qquad 2-2v-7v^2 = \dfrac{C}{x^2}$.

Rather than try to solve for $v$ in this case (b/c it looks messy), perhaps we can go ahead and replace $v = y/x$.

$$\rightarrow \quad 2 - \frac{2y}{x} - \frac{7y^2}{x^2} = \frac{C}{x^2}$$

we are looking for implicit solution $F(x,y) = C$, so we rearrange

$$\rightarrow \quad \boxed{2x^2 - 2xy - 7y^2 = C}$$

Of course, if we needed explicit solution $y = \cdots$, then we would want explicit $v = \cdots$ to sub into $y = vx$. But problem only asked for implicit $F(x,y) = C$..

Bernoulli Eq: $y' + P(x)y = Q(x)y^{n}$ ($n \in \mathbb{R}$)

- $n=0 \longrightarrow y' + Py = Q$ (linear)
- $n=1 \longrightarrow y' + (P-Q)y = 0$ (linear)

or $y' = (Q-P)y$ (~~linear~~ separable)

★ For Bernoulli Eq, use sub $v = y^{1-n}$ ($y = v^{\frac{1}{1-n}}$).

This makes ODE into 1st order linear eq.

Ex: $\{x^2 y' + xy = y^2\}$

Standard form: $y' + \frac{1}{x}y = \frac{1}{x^2}y^{2}$ $n=2$

Set: $v = y^{1-2} = y^{-1} \Longrightarrow \boxed{y = v^{-1}}$

$\Longrightarrow \boxed{y' = -v^{-2}v'}$

sub: $-v^{-2}(v') + \frac{1}{x}v^{-1} = \frac{1}{x^2}v^{-2}$

$\rightarrow \boxed{v' - \frac{1}{x}v = \frac{-1}{x^2}}$ (1st order linear)

(Read rest of example in notes) ?

---

↳ (cont.) $v' + Pv = Q$ 1st order linear, $P = -\frac{1}{x}$, $Q = \frac{-1}{x^2}$

So $\rho = e^{\int P dx} \longrightarrow e^{\int -1/x \, dx} = e^{-\ln x} = x^{-1}$.

Complement: $\rho \cancel{y}v' + \rho' v = \rho Q \longrightarrow \rho v = \int \rho Q dx + C$.

$\rightarrow \frac{1}{x}v = \int -\frac{1}{x^3}dx + C \Rightarrow \frac{1}{x}v = \frac{1}{2}x^{-2} + C$

$\Longrightarrow \boxed{v = \frac{1}{2}x^{-1} + Cx = \frac{1}{2x} + Cx.}$.

fine, but can clean out fractions from denom.

"unsub": $y = v^{-1} \Longrightarrow \boxed{y = \frac{1}{\frac{1}{2x} + Cx}} \cdot \frac{2x}{2x} \longrightarrow \boxed{y = \frac{2x}{1 + Cx^2}}$

$\underline{Ex}$ (homogeneous) Find a general solution for

$$x^2 y' = xy + x^2 e^{y/x}$$

---

First isolate $y' = y/x + e^{y/x}$. Powers of $x, y$ look "balanced", let's check if eq is homogeneous: $= \frac{\cancel{c}dy}{\cancel{c}dx}$

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x} \longrightarrow \frac{d(cy)}{d(cx)} = \frac{\cancel{c}y}{\cancel{c}x} + e^{\cancel{c}y/\cancel{c}x} \longrightarrow \frac{dy}{dx} = \frac{y}{x} + e^{y/x},$$

so it's homogeneous, yes.

Thus $\underline{\text{set}}$ $v = y/x \longleftrightarrow \boxed{y = vx}$

$\underline{\text{Impl. diff}}$ $\Longrightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot \cancel{\frac{dx}{dx}}^{=1}$ (alt: $y' = v'x + v(x)'$, $y' = v'x + v$)

$$\boxed{\frac{dy}{dx} = x\frac{dv}{dx} + v}$$

$\underline{\text{Sub.}}$ : $\frac{dy}{dx} = \frac{y}{x} + e^{y/x} \longrightarrow x\frac{dv}{dx} + v = v + e^v$

$$\longrightarrow \boxed{x\frac{dv}{dx} = e^v} \quad \text{(separable)}$$

$\underline{\text{Solve for } v}$ : $\int e^{-v} dv = \int \frac{dx}{x}$

$$-e^{-v} = \ln|x| + C$$

$$-v = \ln(C - \ln|x|)$$

$$\underline{v = -\ln(C - \ln|x|)}$$

"unsub" : $\boxed{y = vx = -x\ln(C - \ln|x|)}$